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Fractional Derivative of Arbitrary Real Power of Fractional Analytic Function

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Abstract: **In this paper, we obtain the fractional derivative formula of any real power of fractional analytic function based on a new multiplication of fractional analytic functions and Jumarie's modified Riemann-Liouville (R-L) fractional derivative. The new multiplication we defined is a natural operation of fractional analytic functions. The chain rule for fractional derivatives plays an important role in this article. In addition, we give some examples to illustrate this formula.**

Keywords: **Fractional analytic function, New multiplication, Jumarie's modified R-L fractional derivative, Chain rule for fractional derivatives.**

I. INTRODUCTION

Fractional calculus is different from classical calculus. There is no unique definition of fractional derivative and integral. Commonly used definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative and Jumarie's modified R-L fractional derivative [1-4]. Fractional calculus is a field of mathematical analysis. It studies and applies the integral and derivative of arbitrary order. In recent years, fractional calculus has been widely used in physics, engineering, economics, and other fields [5-8].

In this article, based on a new multiplication of fractional analytic functions and Jumarie type of modified R-L fractional derivative, the fractional derivative formula of any real power of fractional analytic functions is obtained. The chain rule for fractional derivatives plays an important role in this study. In fact, our formula is similar to the traditional derivative formula of arbitrary real power of analytical function. Moreover, the new multiplication we defined is a natural operation of fractional analytic functions. In addition, we also give some examples to illustrate the formula we obtained.

II. DEFINITIONS AND PROPERTIES

The following is the fractional calculus used in this paper.

Definition 2.1 ([9])**:** Suppose that $0 < \alpha \le 1$, and a is a real number. The Jumarie type of modified Riemann-Liouville α-fractional derivative and integral are respectively defined as follows:

$$\left({}_{a}D_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{a}^{\theta}\frac{f(x)-f(a)}{(\theta-x)^{\alpha}}dx\,, \tag{1}$$

$$\left({}_{a}I_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(\alpha)}\int_{a}^{\theta}\frac{f(x)}{(\theta-x)^{1-\alpha}}dx\,, \tag{2}$$

where $\Gamma(y) = \int_0^{+\infty} t^{y-1}e^{-t}\,dt$ is the gamma function.

Proposition 2.2 ([10])**:** *Let* α, β, c *be real constants and* $\beta \ge \alpha > 0$, *then*

$$\left({}_{0}D_{\theta}^{\alpha}\right)[\theta^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\theta^{\beta-\alpha}, \tag{3}$$

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and

$$\left({}_{0}D_{\theta}^{\alpha}\right)[c]=0 \quad . \tag{4}$$

The following is the definition of fractional analytic function.

Definition 2.3 ([12])**:** Let , θ_0 , and a_k be real numbers for all k, $\theta_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval (θ_0-s,θ_0+s), then we say that $f_\alpha(\theta^\alpha)$ is α-fractional analytic at θ_0, where s is the radius of convergence about θ_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we give a new multiplication of fractional analytic functions.

Definition 2.4 ([11])**:** If $0<\alpha\leq 1$, $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \tag{6}$$

Then

$$\begin{aligned}&f_\alpha(\theta^\alpha)\otimes g_\alpha(\theta^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\theta^{k\alpha}.\end{aligned} \tag{7}$$

That is,

$$\begin{aligned}&f_\alpha(\theta^\alpha)\otimes g_\alpha(\theta^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}.\end{aligned} \tag{8}$$

In the following, the arbitrary power of fractional analytic function is defined.

Definition 2.5: Let $0<\alpha\leq 1$ and r be any real number. The r-th power of the α-fractional analytic function $f_\alpha(\theta^\alpha)$ is defined by

$$[f_\alpha(\theta^\alpha)]^{\otimes r}=E_\alpha\left(rLn_\alpha\left(f_\alpha(\theta^\alpha)\right)\right). \tag{9}$$

Definition 2.6 ([9])**:** Let $0<\alpha\leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ be α-fractional analytic functions,

$$f_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}\theta^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are defined as follows:

$$(f_\alpha\circ g_\alpha)(\theta^\alpha)=f_\alpha\left(g_\alpha(\theta^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(\theta^\alpha)\right)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha\circ f_\alpha)(\theta^\alpha)=g_\alpha\left(f_\alpha(\theta^\alpha)\right)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(\theta^\alpha)\right)^{\otimes k}. \tag{13}$$

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Definition 2.7 ([9])**:** Let $0 < \alpha \le 1$. If $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = (g_\alpha \circ f_\alpha)(\theta^\alpha) = \frac{1}{\Gamma(\alpha+1)}\theta^\alpha. \tag{14}$$

Then these two fractional analytic functions are called inverse to each other.

In the following, we introduce some fractional analytic functions.

Definition 2.8 ([9, 12])**:** Let $0 < \alpha \le 1$, and θ be a real variable. The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \tag{15}$$

The α-fractional logarithmic function $Ln_\alpha(\theta^\alpha)$ is the inverse function of the $E_\alpha(\theta^\alpha)$. Furthermore, the α-fractional sine and cosine function are defined as follows:

$$sin_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{16}$$

and

$$cos_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k\alpha}}{\Gamma(2k\alpha+1)}. \tag{17}$$

In addition,

$$sec_\alpha(\theta^\alpha) = \left(cos_\alpha(\theta^\alpha)\right)^{\otimes -1} \tag{18}$$

is called the α-fractional secant function.

$$csc_\alpha(\theta^\alpha) = \left(sin_\alpha(\theta^\alpha)\right)^{\otimes -1} \tag{19}$$

is the α-fractional cosecant function.

$$tan_\alpha(\theta^\alpha) = sin_\alpha(\theta^\alpha) \otimes sec_\alpha(\theta^\alpha) \tag{20}$$

is the α-fractional tangent function. And

$$cot_\alpha(\theta^\alpha) = cos_\alpha(\theta^\alpha) \otimes csc_\alpha(\theta^\alpha) \tag{21}$$

is the α-fractional cotangent function.

Theorem 2.9 ([12])**:** *Let* $0 < \alpha \le 1$, *then*

$$\left({}_0D_\theta^\alpha\right)[sin_\alpha(\theta^\alpha)] = cos_\alpha(\theta^\alpha), \tag{22}$$

$$\left({}_0D_\theta^\alpha\right)[cos_\alpha(\theta^\alpha)] = -sin_\alpha(\theta^\alpha), \tag{23}$$

$$\left({}_0D_\theta^\alpha\right)[tan_\alpha(\theta^\alpha)] = \left(sec_\alpha(\theta^\alpha)\right)^{\otimes 2}, \tag{24}$$

$$\left({}_0D_\theta^\alpha\right)[cot_\alpha(\theta^\alpha)] = -\left(csc_\alpha(\theta^\alpha)\right)^{\otimes 2}, \tag{25}$$

$$\left({}_0D_\theta^\alpha\right)[sec_\alpha(\theta^\alpha)] = sec_\alpha(\theta^\alpha) \otimes tan_\alpha(\theta^\alpha), \tag{26}$$

$$\left({}_0D_\theta^\alpha\right)[csc_\alpha(\theta^\alpha)] = -csc_\alpha(\theta^\alpha) \otimes cot_\alpha(\theta^\alpha). \tag{27}$$

Theorem 2.10 (chain rule for fractional derivatives) ([9])**:** *Suppose that* $0 < \alpha \le 1$, *and let* $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ *be* α-*fractional analytic functions. Then*

$$\left({}_0D_\theta^\alpha\right)\left[f_\alpha\left(g_\alpha(\theta^\alpha)\right)\right] = \left({}_0D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)]\left(g_\alpha(\theta^\alpha)\right) \otimes \left({}_0D_\theta^\alpha\right)[g_\alpha(\theta^\alpha)]. \tag{28}$$

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III. MAIN RESULT AND EXAMPLES

In this section, we obtain the fractional derivative of arbitrary real power of fractional analytic function. Moreover, we give some examples to illustrate our result.

Theorem 3.1: *If* $0 < \alpha \leq 1$ *, and* r *is a real number. Then the* α*-fractional derivative of* r*-th power of the* α*-fractional analytic function* $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}$ *is*

$$\left({}_0D_\theta^\alpha\right)[[f_\alpha(\theta^\alpha)]^{\otimes r}] = r[f_\alpha(\theta^\alpha)]^{\otimes(r-1)} \otimes \left({}_0D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)]. \tag{29}$$

Proof By chain rule for fractional derivatives,

$$\left({}_0D_\theta^\alpha\right)[[f_\alpha(\theta^\alpha)]^{\otimes r}]$$

$$= \left({}_0D_\theta^\alpha\right)\left[E_\alpha\left(rLn_\alpha(f_\alpha(\theta^\alpha))\right)\right]$$

$$= E_\alpha\left(rLn_\alpha(f_\alpha(\theta^\alpha))\right) \otimes \left({}_0D_\theta^\alpha\right)[rLn_\alpha(f_\alpha(\theta^\alpha))]$$

$$= [f_\alpha(\theta^\alpha)]^{\otimes r} \otimes r[f_\alpha(\theta^\alpha)]^{\otimes -1} \otimes \left({}_0D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)]$$

$$= r[f_\alpha(\theta^\alpha)]^{\otimes(r-1)} \otimes \left({}_0D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)].$$ ■

Example 3.2: Assume that $0 < \alpha \leq 1$, and r is a real number. Then by Theorems 2.9, 2.10, and 3.1, we obtain

$$\left({}_0D_\theta^\alpha\right)[[sin_\alpha(\theta^\alpha)]^{\otimes r}] = r[sin_\alpha(\theta^\alpha)]^{\otimes(r-1)} \otimes cos_\alpha(\theta^\alpha), \tag{30}$$

$$\left({}_0D_\theta^\alpha\right)[[cos_\alpha(\theta^\alpha)]^{\otimes r}] = -r[cos_\alpha(\theta^\alpha)]^{\otimes(r-1)} \otimes sin_\alpha(\theta^\alpha), \tag{31}$$

$$\left({}_0D_\theta^\alpha\right)[[tan_\alpha(\theta^\alpha)]^{\otimes r}] = r[tan_\alpha(\theta^\alpha)]^{\otimes(r-1)} \otimes \left(sec_\alpha(\theta^\alpha)\right)^{\otimes 2}, \tag{32}$$

$$\left({}_0D_\theta^\alpha\right)[[cot_\alpha(\theta^\alpha)]^{\otimes r}] = -r[cot_\alpha(\theta^\alpha)]^{\otimes(r-1)} \otimes \left(csc_\alpha(\theta^\alpha)\right)^{\otimes 2}, \tag{33}$$

$$\left({}_0D_\theta^\alpha\right)[[sec_\alpha(\theta^\alpha)]^{\otimes r}] = r[sec_\alpha(\theta^\alpha)]^{\otimes r} \otimes tan_\alpha(\theta^\alpha), \tag{34}$$

$$\left({}_0D_\theta^\alpha\right)[[csc_\alpha(\theta^\alpha)]^{\otimes r}] = -r[csc_\alpha(\theta^\alpha)]^{\otimes r} \otimes cot_\alpha(\theta^\alpha). \tag{35}$$

$$\left({}_0D_\theta^\alpha\right)\left[\left[E_\alpha\left(sin_\alpha(\theta^\alpha)\right)\right]^{\otimes r}\right] = r\left[E_\alpha\left(sin_\alpha(\theta^\alpha)\right)\right]^{\otimes r} \otimes cos_\alpha(\theta^\alpha)\,. \tag{36}$$

IV. CONCLUSION

Based on Jumarie type modified R-L fractional derivative and a new multiplication of fractional analytic functions, the fractional derivative formula of any real power of fractional analytic function is obtained in this paper. In fact, this formula is a natural generalization of the formula of traditional derivative of real power of analytic functions. On the other hand, the chain rule for fractional derivatives plays an important role in this article. The new multiplication we defined is a natural operation of fractional analytic functions. In the future, we will use this formula and the new multiplication to expand the research topics to the problems of applied mathematics and fractional calculus.

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